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Best One-Sided L^1-Approximation by Blending Functions of Order (2, 2)

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Let $f \in C^{2,2}([-1,1]^2)$ be a real function satisfying $\partial^4 f/\partial x^2\,\partial y^2 \geqslant 0$ on $[-1,1]^2$. We study the problem of best one-sided L^1-approximation to f from the linear space $\{h \in C^{2,2}([-1,1]^2) : \partial^4 h/\partial x^2\,\partial y^2 = 0\}$ of all blending functions of order (2, 2). The unique best one-sided L^1-approximant to f from above is characterized by transfinite Hermite interpolation on the canonical grid $\{(x,y) \in [-1,1]^2 : |x| = |y|\}$. For f even with respect to one of its variables we characterize the unique best one-sided L^1-approximant to f from below by transfinite Hermite interpolation on the canonical grid $\{(x,y) \in [-1,1]^2 : |x|+|y| = 1\}$. There is no canonical grid for the entire cone class of functions f with $\partial^4 f/\partial x^2\,\partial y^2 \geqslant 0$ on $[-1,1]^2$ when we approximate from below. The best one-sided L^1-approximant from above has the smoothness of f. The best one-sided L^1-approximant to f from below is a *blending-spline* function with *two line segment knots* $\{(x,0) : -1 \leqslant x \leqslant 1\}$ and $\{(0,y): -1 \leqslant y \leqslant 1\}$; i.e., the best one-sided approximation to f from below possesses a saturation effect with respect to the smoothness of f. © 2002 Elsevier Science (USA)

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1. INTRODUCTION AND MAIN RESULTS

The classical univariate algebraic polynomials have a natural multivariate extension by the so-called *blending functions*. The real linear space $B^{m,n}(I^2)$ of all blending functions of order (m, n) is defined as

$$B^{m,n}(I^2) := \left\{ h \in C^{m,n}(I^2) : D^{m,n}h := \frac{\partial^{m+n}}{\partial x^m\, \partial y^n} h = 0 \text{ on } I^2 \right\},$$

where $I^2 := [-1, 1]^2$ and

$$C^{m,n}(I^2) := \{ f : I^2 \to \mathbf{R} : D^{i,j} f \in C(I^2),\, 0 \leqslant i \leqslant m,\, 0 \leqslant j \leqslant n \}.$$

The linear space $B^{m,n} := B^{m,n}(I^2)$ is infinite-dimensional in contrast to the fact that the linear space of all univariate algebraic polynomials of degree less than or equal to $m-1$,

$$\pi_m := \left\{ p \in C^m(I) : \frac{d^m}{dx^m} p = 0 \right\},$$

is of finite dimension m.

Any blending function $h \in B^{m,n}$ can be represented in the form

$$h(x, y) = \sum_{k=0}^{m-1} a_k(y)\, x^k + \sum_{l=0}^{n-1} b_l(x)\, y^l, \qquad a_k \in C^n(I), \quad b_l \in C^m(I).$$

Note that *this representation is not unique*. Conversely, each function of this form is a $B^{m,n}$-blending function, so

$$B^{m,n} = \left\{ h \in C^{m,n}(I^2) : h(x, y) = \sum_{k=0}^{m-1} a_k(y)\, x^k + \sum_{l=0}^{n-1} b_l(x)\, y^l \right\}.$$

The blending functions are an attractive tool for multivariate approximation due to their wide applications in various mathematical fields such as numerical methods for partial differential equations, cubature formulae for approximate integration of multidimensional integrals, computer aided geometric design, etc. (see [7, 17] and the references there).

In various cases the explicit construction of best L^1-approximants can be given in terms of interpolation with respect to *a canonical point set (grid)*. Then if a canonical point set exists, the nonlinear problem of best (one-sided) L^1-approximation becomes a linear one for a (sufficiently large) class of functions. Appropriately chosen (transfinite) interpolation on the canonical grid yields explicit characterization of the best approximants that creates a basis for an algorithmic approach and numerical methods. This makes the approximation via *interpolation on canonical grids* useful in various fields of applied mathematics, physics and engineering.

As a classical example we mention the well known *Markov theorem* [1, pp. 82–85; 11, p. 87] on best L^1-approximation by polynomials. The best L^1-approximation has been subject to much research activity during the last years when dealing with finite dimensional approximating space (polynomials, Chebyshev systems) [25].

On the other hand there are recent characterization results for *multivariate L^1-approximants* by *transfinite interpolation* with respect to *canonical point sets* where the approximating space is *infinite-dimensional.* For example if we approximate subharmonic functions by harmonic functions on the unit ball then a concentric sphere is a canonical point set [2]. If one considers approximation by blending functions on the unit square then the canonical point set forms a so-called *blending grid* consisting of appropriate horizontal and vertical lines [20]. For further references concerning characterization of best multidimensional L^1-approximants by canonical point sets the reader may consult the survey papers [5, 21]. Approximation by $B^{1,1}$-blending functions in L^1-norm has been considered in [22].

In [3, 4] the authors characterize best one-sided L^1-approximation to a continuous function by harmonic functions in terms of the set (usually called zero-set) where a best one-sided approximant coincides with (*interpolates*) the function under approximation. As they mentioned *the best one-sided L^1-approximant by harmonic functions may not exist and may be very far from unique.*

In [9] it has been shown that *a canonical blending grid of vertical and horizontal lines* occurs in the case of *one-sided L^1-approximation* by blending functions, too, but *only when we approximate by a restricted, appropriately chosen, subclass of blending functions.* Moreover, an example was given, showing that this is not the case when we approximate by the entire linear space $B^{m,n}$.

Recently, we have found the existence of a canonical grid for best one-sided L^1-approximation when we approximate by *the entire infinite-dimensional space of $B^{1,1}(I^2)$-blending functions* to the cone class $\{f: D^{1,1}f \geqslant 0 \text{ on } I^2\}$.

THEOREM A [14]. *Let*

$$\varDelta^* := \{(x, y) \in I^2 : x = y\} \qquad \textit{and} \qquad \varDelta_* := \{(x, y) \in I^2 : x = -y\}$$

be the diagonals of I^2. *Let* $f \in C^{1,1}(I^2)$ *satisfy* $D^{1,1}f \geqslant 0$ *on* I^2. *Then we have*

(a) *the unique solution*

$$h^*(x, y) := f(-1, -1) + \int_{-1}^{x} D^{1,0}f(t, t)\, dt + \int_{-1}^{y} D^{0,1}f(t, t)\, dt$$

of the transfinite Hermite interpolation problem

$$h^*_{|\Delta^*} = f_{|\Delta^*} \qquad \text{and} \qquad \mathbf{grad}\, h^*_{|\Delta^*} = \mathbf{grad}\, f_{|\Delta^*} \qquad (h^* \in B^{1,1})$$

is the unique best one-sided L^1*-approximant to* f *from above with respect to* $B^{1,1}$*;*

(b) *the unique solution*

$$h_*(x, y) := f(-1, 1) + \int_{-1}^{x} D^{1,0}f(t, -t)\, dt - \int_{-1}^{-y} D^{0,1}f(t, -t)\, dt$$

of the transfinite Hermite interpolation problem

$$h_{*|\Delta_*} = f_{|\Delta_*} \qquad \text{and} \qquad \mathbf{grad}\, h_{*|\Delta_*} = \mathbf{grad}\, f_{|\Delta_*} \qquad (h_* \in B^{1,1})$$

is the unique best one-sided L^1*-approximant to* f *from below with respect to* $B^{1,1}$.

The method in [14] has been extended in [15] to give a solution in the case of a best one-sided L^1-approximation from *the infinite-dimensional space of* $B^{2,1}(I^2)$*-blending functions* to the cone class $\{f \colon D^{2,1}f \geqslant 0 \text{ on } I^2\}$. The characterization of the best one-sided L^1-approximants is given by transfinite Hermite interpolation on *the canonical point set* $\bigvee^* := \{(x, y) \in I^2 \colon y = 2\,|x| - 1\}$ when we approximate from above, resp. on *the canonical set* $\bigwedge^* := \{(x, y) \in I^2 : y = -2\,|x| + 1\}$ in the case of approximation from below. Analogous results hold when we approximate onesidedly by $B^{1,2}(I^2)$-blending functions.

The problem of existence of a best uniform $B^{1,1}$-approximant to a given function has been studied in [24, Chap. 2]. The basic notions when the author deals with this problem is *the lightning bolt* that gives rise to a multidimensional extension of the Chebyshev alteration characterization of best uniform approximation by polynomials. To the best of our knowledge the notion of lightning bolt appears for the first time in the work of V. I. Arnold [6] where the thirteenth Hilbert problem has been solved. Algorithmic approaches to best uniform $B^{1,1}$-approximants are presented in [13, 18]. In contrast to Theorem A the results in [16, 23] say the following:

THEOREM B. *Let $f \in C^{1,1}(I^2)$ and $D^{1,1}f \geqslant 0$ on I^2. Then the best uniform approximant to f from $B^{1,1}$ is never unique if $D^{1,1}f(x_o, y_o) > 0$ at some point $(x_o, y_o) \in I^2$.*

The theorems A and B show how the results in multidimensional approximation theory (when dealing with infinite-dimensional approximating space too) could differ from the corresponding ones in the one-dimensional case.

In the case of best one-sided L^1-approximation from above our main result is the following

THEOREM 1.1. *Let $f \in C^{2,2}(I^2)$ and $D^{2,2}f \geqslant 0$ on I^2. Then the unique best one-sided L^1-approximant h_f^* from $\{h \in B^{2,2}(I^2) : h \geqslant f\}$ to f is characterized by the following transfinite Hermite interpolation conditions*

$$h^*_{f|\times^*} = f_{|\times^*} \qquad \text{and} \qquad \mathbf{grad}\, h^*_{f|\times^*} = \mathbf{grad}\, f_{|\times^*}$$

on the canonical grid $\times^* := \{(x, y) \in I^2 : |x| = |y|\}$.

In the case of best one-sided L^1-approximation from below the situation is different. There is no canonical grid for the entire cone class $f \in C^{2,2}(I^2)$ and $D^{2,2}f \geqslant 0$ on I^2 (see Proposition 3.4). However, for the subcone of functions which are even with respect to one of its variables a canonical grid exists.

THEOREM 1.2. *Let $f \in C^{2,2}(I^2)$ and $D^{2,2}f \geqslant 0$ on I^2. In addition, let $f = f_0 + h$, where f_0 is an even function with respect to one of its variables and $h \in B^{2,2}(I^2)$. Then the unique best one-sided L^1-approximant h_*^f to f from below by the infinite-dimensional linear space $B^{2,2}(I^2)$ is characterized via the transfinite Hermite interpolation conditions*

$$h^f_{*|\diamond_*} = f_{|\diamond_*} \qquad \text{and} \qquad \mathbf{grad}\, h^f_{*|\diamond_*} = \mathbf{grad}\, f_{|\diamond_*}$$

on the canonical grid $\diamond_* := \{(x, y) \in I^2 : |x| + |y| = 1\}$.

Remark. The real linear space $B^{m,n}(I^2)$ is a bivariate analogue of the univariate space of algebraic polynomials of a certain degree. In view of this, the best approximants h_f^* and h_*^f can be seen as *extended $B^{2,2}$ polynomials*. Thus, Theorem 1.1 and Theorem 1.2 can be considered as a natural bivariate blending-type extension of classical one-dimensional results on best one-sided L^1-approximation by polynomials (Chebyshev systems) [8, 12].

Remark. Let us mention also that the best one-sided L^1-approximant from above h_f^* has the smoothness of f. For example, if f is a polynomial then h_f^* is a polynomial, too. On the other hand, the best one-sided L^1-approximant h_*^f to f from below is a $B^{2,2}(I^2)$*-spline function with two line segment knots* $\{(x, 0)\colon -1 \leqslant x \leqslant 1\}$ and $\{(0, y)\colon -1 \leqslant y \leqslant 1\}$. In other words, the best one-sided from below approximant h_*^f possesses a saturation effect with respect to the smoothness of f.

EXAMPLE. Let $f(x, y) = x^2y^2$. Then $h_f^*(x, y) = (x^4 + y^4)/2$. However,

$$h_*^f(x, y) = \tfrac{1}{2}(x^4 + y^4) - \tfrac{4}{3}(|x|^3 + |y|^3) + (x^2 + y^2) - \tfrac{1}{6}$$

and evidently, $h_*^f \notin C^{3,0}(I^2) \cup C^{0,3}(I^2)$.

Notation. Let κ be a point set in I^2 and let $g \in C^{2,2}(I^2)$. Let $g_{|\kappa}$ be the restriction (the trace) of the function g on the point set κ. Similarly, let $\partial f/\partial x_{|\kappa}$ and $\partial f/\partial y_{|\kappa}$ be the restrictions (the traces) of $\frac{\partial f}{\partial x}$, resp. $\frac{\partial f}{\partial y}$ on the point set κ. Then $g_{|\kappa} = 0$ stands for the fact that g is identically zero on κ. Analogously, we symbolize the fact that $\partial f/\partial x_{|\kappa}$ and $\partial f/\partial y_{|\kappa}$ are both identically zero by $\mathbf{grad}\ g_{|\kappa} = \mathbf{0}$.

The following *Maximum/Minimum principles* follow from Theorem 2.3, respectively Lemma 3.2. They are useful in the proofs of Theorem 1.1 and Theorem 1.2. These principles are not maximum/minimum principles in the proper classic sense. They refer to subsets in I^2 rather than to the boundary of the given set as it is in the case of the Laplace differential operator (subharmonic functions). However, these principles express the same idea in a more general form. Namely, under some suppositions for a given function on I^2 we can localize considerably the point set where the function attains maximum/minimum on I^2.

Maximum Principle. Let $g(x, y) \in C^{2,2}(I^2)$ satisfy

$$g_{|\times^*} = 0, \quad \mathbf{grad}\ g_{|\times^*} = \mathbf{0}, \qquad \text{and} \qquad D^{2,2}g > 0 \text{ on } I^2.$$

Then g cannot attain a maximum on $I^2 \setminus \times^* := \{(x, y) \in I^2 : |x| \neq |y|\}$. In other words $g(x, y) < 0$ on $I^2 \setminus \times^*$.

Minimum Principle. Let a function $g(x, y) \in C^{2,2}(I^2)$ satisfy

$$g_{|\diamondsuit_*} = 0, \qquad \mathbf{grad}\ g_{|\diamondsuit_*} = \mathbf{0}, \qquad \text{and} \qquad D^{2,2}g > 0 \text{ on } I^2.$$

Then g cannot attain a minimum on $\mathbf{int}\,\diamondsuit_* := \{(x, y) \in I^2 : |x| + |y| < 1\}$. In other words $g(x, y) > 0$ on $\mathbf{int}\,\diamondsuit_*$.

2. BEST ONE-SIDED L^1-APPROXIMATION FROM ABOVE BY BLENDING FUNCTIONS OF ORDER (2, 2)

2.1. *Transfinite Interpolation by $B^{2,2}(I^2)$-Blending Functions*

In this subsection we study the problem of existence and uniqueness of a blending transfinite Hermite interpolant $h^* \in B^{2,2}(I^2)$ that interpolates a given function $f \in C^{2,2}(I^2)$ on the point set (grid)

$$\times^* := \{(x, y) \in I^2 : |x| = |y|\}$$

by the interpolation conditions

$$h^*_{|\times^*} = f_{|\times^*} \qquad \text{and} \qquad \mathbf{grad}\, h^*_{|\times^*} = \mathbf{grad}\, f_{|\times^*}. \tag{2.1}$$

Let $h \in B^{2,2}(I^2)$. Then h can be represented in a form

$$h(x, y) = a(y)\, x + b(y) + c(x)\, y + g(x), \tag{2.2}$$

where $a(\cdot), b(\cdot), c(\cdot), g(\cdot) \in C^2(I)$, $I := [-1, 1]$. Note, that the representation (2.2) is not unique.

First we construct a $B^{2,2}(I^2)$-blending function h_1 that interpolates f by the interpolation conditions

$$h_{1|I_1} = f_{|I_1}; \qquad \mathbf{grad}\, h_{1|I_1} = \mathbf{grad}\, f_{|I_1}, \tag{2.3}$$

where $I_1 := \{(x, y) \in I^2 : x = y\}$. All $B^{2,2}(I^2)$-functions of the form

$$h_1(x, y) = a(y)\, x + c(x)\, y + \left[\int_0^x f^{(1,0)}(t, t)\, dt - \int_0^x a(t)\, dt - \int_0^x t c'(t)\, dt\right]$$
$$+ \left[\int_0^y f^{(0,1)}(t, t)\, dt - \int_0^y c(t)\, dt - \int_0^y t a'(t)\, dt\right] + f(0, 0),$$

where $a(\cdot), c(\cdot) \in C^2(I)$, describe the solution of the interpolation problem (2.3). Analogously, for the point set $I_2 := \{(x, y) \in I^2 : x = -y\}$ all the solutions of the interpolation problem

$$h_{2|I_2} = f_{|I_2}; \qquad \mathbf{grad}\, h_{2|I_2} = \mathbf{grad}\, f_{|I_2}; \qquad h_2 \in B^{2,2}(I^2) \tag{2.4}$$

are given by the $B^{2,2}(I^2)$-blending functions of the form

$$h_2(x, y) = a(y)\, x + c(x)\, y + \left[\int_0^x f^{(1,0)}(t, -t)\, dt - \int_0^x a(-t)\, dt + \int_0^x t c'(t)\, dt\right]$$
$$+ \left[\int_0^y f^{(0,1)}(-t, t)\, dt - \int_0^y c(-t)\, dt + \int_0^y t a'(t)\, dt\right] + f(0, 0),$$

for arbitrary $a(\cdot), c(\cdot) \in C^2(I)$. Note, that h_1 and h_2 are sums of $a(y)\,x+c(x)\,y$ with $B^{1,1}$-blending functions that interpolate $f(x, y)-a(y)\,x-c(x)\,y$ on the diagonals I_1 and I_2, respectively (see Theorem A).

Denote by S, S_1 and S_2 the sets of all solutions of the interpolation problems (2.1), (2.3) and (2.4), respectively. It is obvious that $S=S_1 \cap S_2$. In other words, to find the set S we have to determine *the non-unique basic functions* $a(\cdot), c(\cdot) \in C^2(I)$ in both interpolation problems (2.3) and (2.4) such that

$$h^*(x, y) := h_1(x, y) \equiv h_2(x, y) \in C^{2,2}(I^2).$$

In our further considerations we shall use the notations

$$\Omega_1 := \{(x, y) \in I^2 : x>0,\ y>0\}; \qquad \Omega_2 := \{(x, y) \in I^2 : x<0,\ y<0\};$$
$$\Omega_3 := \{(x, y) \in I^2 : x<0,\ y>0\}; \qquad \Omega_4 := \{(x, y) \in I^2 : x>0,\ y<0\};$$

and

$$I_{x,0} := \{(x, y) \in I^2 : y=0\}; \qquad I_{0,y} := \{(x, y) \in I^2 : x=0\}.$$

Obviously, $I_{x,0} \cup I_{0,y} \cup (\bigcup_{i=1}^4 \Omega_i)=I^2$. All solutions $h^*(x, y)$ of the interpolation problem (2.1) are determined by

$$h^*(x, y)=\begin{cases} h_1(x, y) & \text{on } \ \Omega_1 \cup \Omega_2 \\ h_2(x, y) & \text{on } \ \Omega_3 \cup \Omega_4, \end{cases}$$

where the $B^{2,2}$-blending functions $h_1(x, y)$ and $h_2(x, y)$ satisfy the transfinite interpolation (matching) conditions

$$D^{i,j}h_{1\,|\,I_{x,0}}=D^{i,j}h_{2\,|\,I_{x,0}},\ D^{i,j}h_{1\,|\,I_{0,y}}=D^{i,j}h_{2\,|\,I_{0,y}}, \qquad 0 \leqslant i,\ j \leqslant 2.$$

Note, that this approach will give all solutions of the interpolation problem (2.1).

These matching conditions are equivalent to the following system of integral-differential equations (interpolation conditions):

$$\text{(1)}\quad 2\int_0^y ta'(t)\,dt+\int_0^y [c(t)-c(-t)]\,dt=\int_0^y [f^{(0,1)}(t, t)-f^{(0,1)}(-t, t)]\,dt;$$

$$\text{(2)}\quad 2\int_0^x tc'(t)\,dt+\int_0^x [a(t)-a(-t)]\,dt=\int_0^x [f^{(1,0)}(t, t)-f^{(1,0)}(t, -t)]\,dt;$$

$$\text{(3)}\quad 2ya'(y)+c(y)-c(-y)=f^{(0,1)}(y, y)-f^{(0,1)}(-y, y);$$

$$\text{(4)}\quad 2xc'(x)+a(x)-a(-x)=f^{(1,0)}(x, x)-f^{(1,0)}(x, -x),$$

where the functions $a(\cdot)$ and $c(\cdot) \in C^2(I)$. The matching conditions (3) and (4) follow directly from (1) and (2).

We shall use the above considerations in the proof of the uniqueness of the solution of the interpolation problem (2.1).

LEMMA 2.1. *Suppose that the interpolation problem* (2.1) *has a solution. Then this solution is unique.*

Proof. It is sufficient to prove that the transfinite interpolation problem (2.1) with zero interpolation conditions has the constant zero as a unique solution. From (3) and (4) we obtain that the basic functions $a(\cdot)$, $c(\cdot) \in C^2(I)$ satisfy the system

$$\left| \begin{array}{l} 2xa'(x)+c(x)-c(-x)=0 \\ 2xc'(x)+a(x)-a(-x)=0. \end{array} \right. \tag{2.5}$$

We have $a'(x)+a'(-x)=[c(-x)-c(x)]/x$, $2xc''(x)+2c'(x)+a'(x)+a'(-x)=0 \Rightarrow 2xc''(x)+2c'(x)+[c(-x)-c(x)]/x=0$ or $[x(c'(x)-c'(-x))]'=0$. Hence, $x[c'(x)-c'(-x)]=0$ so, $[c(x)+c(-x)]'=0$. Thus $c(x)+c(-x)=2c(0)$ and $c'(x)=c'(-x)$ and $xa'(x)=-c(x)+c(0)$. Analogously, $xc'(x)=-a(x)+a(0)$. On the other hand $x[a(x)+c(x)]'+[a(x)+c(x)]=a(0)+c(0)$ and from here $a(x)+c(x)=a(0)+c(0)$. Now by using that $xa'(x)=-c(x)+c(0)$ we get $[(a(x)-a(0))/x]'=0$ and similarly, $[(c(x)-c(0))/x]'=0$.

Hence, all solutions of the system (2.5) *are given by linear functions*

$$a(x)=xa'(0)+a(0) \qquad \text{and} \qquad c(x)=xc'(0)+c(0)$$

under the additional condition $a'(0)+c'(0)=0$.

Then for $h_1(x, y)$ we obtain

$$h_1(x, y)=a(0)\,x+c(0)\,y-\int_0^x [ta'(0)+a(0)]\,dt-\int_0^x tc'(0)\,dt$$

$$-\int_0^y [tc'(0)+c(0)]\,dt-\int_0^y ta'(0)\,dt \equiv 0.$$

Analogously, $h_2(x, y) \equiv 0$. From here, we conclude that $h^* \equiv 0$. ∎

In the next lemma we give an explicit expression for the unique solution of the interpolation problem (2.1).

LEMMA 2.2. *For a given function $f \in C^{2,2}(I^2)$ the transfinite interpolation problem* (2.1) *has a unique solution in the infinite-dimensional linear space $B^{2,2}(I^2)$.*

Proof. The uniqueness follows by Lemma 2.1. Any solution of the transfinite interpolation problem (2.1) can be given in the form (2.2), where the functions $c(\cdot), a(\cdot) \in C^2(I^2)$ satisfy the matching conditions (1), (2) $\Rightarrow$ (3), (4). From here we have $h_1 = h_2 := h^*$ and

$$h^*(x, y) = a(y)\, x + c(x)\, y + \tfrac{1}{2} \int_0^x [f^{(1,0)}(t, t) + f^{(1,0)}(t, -t)]\, dt$$

$$+ \tfrac{1}{2} \int_0^y [f^{(0,1)}(t, t) + f^{(0,1)}(-t, t)]\, dt - \tfrac{1}{2} \int_0^x [a(t) + a(-t)]\, dt$$

$$- \tfrac{1}{2} \int_0^y [c(t) + c(-t)]\, dt + f(0, 0). \tag{2.6}$$

It is easily seen that h^*, as it is defined above, satisfies the interpolation conditions (2.1). The existence of a solution h^* will follow if we prove that the system (1), (2) has a solution $c(\cdot), a(\cdot) \in C^2(I)$. Note, that the representation (2.2) of a $B^{2,2}$-blending function is not unique. By the matching condition (4) we have

$$c'(x) = [(f^{(1,0)}(x, x) - f^{(1,0)}(x, -x)) - (a(x) - a(-x))]/(2x).$$

Writing the above formula for $c'(-x)$ we obtain

$$c(x) + c(-x) - 2c(0) = \int_0^x [f^{(1,0)}(t, t) + f^{(1,0)}(-t, -t)$$

$$- f^{(1,0)}(t, -t) - f^{(1,0)}(-t, t)]/(2t)\, dt.$$

By analogy, a similar expression for $a(\cdot)$ holds also

$$a(y) + a(-y) - 2a(0) = \int_0^y [f^{(0,1)}(t, t) + f^{(0,1)}(-t, -t)$$

$$- f^{(0,1)}(-t, t) - f^{(0,1)}(t, -t)]/(2t)\, dt.$$

By using the notation

$$D_{1,0}(x) := \int_0^x [f^{(1,0)}(t, t) + f^{(1,0)}(-t, -t) - f^{(1,0)}(t, -t) - f^{(1,0)}(-t, t)]/(2t)\, dt,$$

and

$$D_{0,1}(y) := \int_0^y [f^{(0,1)}(t, t) + f^{(0,1)}(-t, -t) - f^{(0,1)}(-t, t) - f^{(0,1)}(t, -t)]/(2t)\, dt,$$

we see that the functions $c(\cdot)$ and $a(\cdot)$ satisfy

$$c(x)+c(-x)=D_{1,0}(x)+2c(0) \quad \text{and} \quad a(y)+a(-y)=D_{0,1}(y)+2a(0).$$

From here and in view of (2.6) any solution h^* of the interpolation problem (2.1) has the form

$$\begin{aligned} h^*(x, y) &= [a(y)-a(0)]\,x+[c(x)-c(0)]\,y \\ &\quad +\tfrac{1}{2}\int_0^x [f^{(1,0)}(t, t)+f^{(1,0)}(t, -t)]\,dt \\ &\quad +\tfrac{1}{2}\int_0^y [f^{(0,1)}(t, t)+f^{(0,1)}(-t, t)]\,dt-\tfrac{1}{2}\int_0^x D_{0,1}(t)\,dt \\ &\quad -\tfrac{1}{2}\int_0^y D_{1,0}(t)\,dt+f(0, 0). \end{aligned} \tag{2.7}$$

Observing that any solution h^* of (2.1) satisfies the interpolation conditions

$$h^*(x, x)=f(x, x) \qquad \text{and} \qquad h^*(x, -x)=f(x, -x)$$

we get in view of (2.7)

$$\begin{aligned} f(x, x)-f(x, -x) &= \tfrac{1}{2}\int_{-x}^x [f^{(0,1)}(t, t)+f^{(0,1)}(-t, t)]\,dt \\ &\quad -\tfrac{1}{2}\int_{-x}^x D_{1,0}(t)\,dt+x[a(x)-a(-x)]+2x[c(x)-c(0)]. \end{aligned}$$

Using the matching condition (4) in the above formula we obtain the next explicit formula for $c(\cdot)$:

$$\begin{aligned} \left[\frac{c(x)-c(0)}{x}\right]' &= -\frac{f(x, x)-f(x, -x)}{2x^3}+\frac{1}{2x^2}[f^{(1,0)}(x, x)-f^{(1,0)}(x, -x)] \\ &\quad +\frac{1}{4x^3}\int_{-x}^x [f^{(0,1)}(t, t)+f^{(0,1)}(-t, t)]\,dt \\ &\quad -\frac{1}{4x^3}\int_{-x}^x D_{1,0}(t)\,dt. \end{aligned}$$

A similar formula holds for $a(\cdot)$:

$$\left[\frac{a(y)-a(0)}{y}\right]' = -\frac{f(y,y)-f(-y,y)}{2y^3}+\frac{1}{2y^2}[f^{(0,1)}(y,y)-f^{(0,1)}(-y,y)]$$
$$+\frac{1}{4y^3}\int_{-y}^{y}[f^{(1,0)}(t,t)+f^{(1,0)}(t,-t)]\,dt$$
$$-\frac{1}{4y^3}\int_{-y}^{y}D_{0,1}(t)\,dt.$$

By making use of (2.7) and the above formulas for $a(\cdot)$, $c(\cdot)$ we get the following explicit expression for the unique solution of the interpolation problem (2.1),

$$\begin{aligned} h_f^*(x,y) &\\ &:= xy\,.\,f^{(1,1)}(0,0)+f(0,0)\\ &+xy\,.\left\{\int_0^y (t[f^{(0,1)}(t,t)-f^{(0,1)}(-t,t)]-[f(t,t)-f(-t,t)])/(2t^3)\,dt\right.\\ &+\int_0^x (t[f^{(1,0)}(t,t)-f^{(1,0)}(t,-t)]-[f(t,t)-f(t,-t)])/(2t^3)\,dt\\ &+\int_0^y \frac{1}{4t^3}\int_{-t}^{t}[f^{(1,0)}(v,v)+f^{(1,0)}(v,-v)]\,dv\,dt\\ &+\int_0^x \frac{1}{4t^3}\int_{-t}^{t}[f^{(0,1)}(v,v)+f^{(0,1)}(-v,v)]\,dv\,dt\\ &\left.-\int_0^y \frac{1}{4t^3}\int_{-t}^{t}D_{0,1}(v)\,dv\,dt-\int_0^x \frac{1}{4t^3}\int_{-t}^{t}D_{1,0}(v)\,dv\,dt\right\}\\ &+\frac{1}{2}\int_0^x [f^{(1,0)}(t,t)+f^{(1,0)}(t,-t)]\,dt\\ &+\frac{1}{2}\int_0^y [f^{(0,1)}(t,t)+f^{(0,1)}(-t,t)]\,dt\\ &-\frac{1}{2}\int_0^x D_{0,1}(t)\,dt-\frac{1}{2}\int_0^y D_{1,0}(t)\,dt.\quad \blacksquare \end{aligned} \tag{2.8}$$

Now we are ready to prove our first main result in this section.

THEOREM 2.3. *Let $f \in C^{2,2}(I^2)$ be a given function. Then we have:*

(a) *the function $h_f^*(x, y)$ defined by* (2.8) *is the unique transfinite Hermite interpolant from $B^{2,2}$ satisfying the interpolation conditions* (2.1)*;*

(b) *the following error representation formula holds true,*

$$f(x, y) - h_f^*(x, y) = -\frac{(x^2 - y^2)^2}{8} D^{2,2} f(\xi, \eta), \tag{2.9}$$

where $(\xi, \eta) = (\xi(x, y), \eta(x, y)) \in I^2$.

Proof of (a). The statement (a) follows directly by Lemma 2.1 and Lemma 2.2. Note, that the uniqueness of h_f^* can be obtained also by using the error representation (b).

Proof of (b). For a given $(x_0, y_0) \in I^2 \setminus \times^*$ we consider the auxiliary function

$$\phi(x, y) := f(x, y) - h_f^*(x, y) + \mathbf{c}.(x^2 - y^2)^2,$$

where the constant

$$\mathbf{c} := -\frac{f(x_0, y_0) - h_f^*(x_0, y_0)}{(x_0^2 - y_0^2)^2}$$

is chosen such that $\phi(x_0, y_0) = 0$. Clearly, we have

$$\phi_{|\times^*} = 0 \quad \text{and} \quad \mathbf{grad}\, \phi_{|\times^*} = \mathbf{0}. \tag{2.10}$$

We are going to show that $D^{2,2}\phi$ vanishes at some point on I^2. Suppose to the contrary that $D^{2,2}\phi > 0$ on I^2. The case $D^{2,2}\phi < 0$ on I^2 is a similar one. The next construction is important for the proof. The *mixed divided difference of order* (2, 2) for the function $\phi(x, y)$ with knots $[-t, t, t]$ ($t \in I^2$ and $t \neq 0$) can be represented as

$$\begin{aligned}
&[-t, t, t]_x\, [-t, t, t]_y\, \phi \\
&\quad = [-t, t, t] \left[\frac{\phi^{(1,0)}(t, \cdot)}{2t} - \frac{\phi(t, \cdot) - \phi(-t, \cdot)}{4t^2} \right] \\
&\quad = \frac{1}{4t^2} \left\{ \phi^{(1,1)}(t, t) - \frac{1}{2t} [\phi^{(1,0)}(t, t) - \phi^{(1,0)}(t, -t)] \right\} \\
&\quad\quad - \frac{1}{8t^3} \left\{ \phi^{(0,1)}(t, t) - \frac{1}{2t} [\phi(t, t) - \phi(t, -t)] \right\} \\
&\quad\quad + \frac{1}{8t^3} \left\{ \phi^{(0,1)}(-t, t) - \frac{1}{2t} [\phi(-t, t) - \phi(-t, -t)] \right\}.
\end{aligned}$$

By using the integral representation of univariate divided differences (see [10] for details) and the mean value theorem we obtain

$$[-t, t, t]_x\,[-t, t, t]_y\,\phi = \int_{I^2} B_1(-t, t, t; u)\,B_1(-t, t, t; v)\,D^{2,2}f(u, v)\,du\,dv$$
$$= \frac{1}{4}\,D^{2,2}\phi(\tilde{\xi}, \tilde{\eta}) > 0, \qquad (\tilde{\xi}, \tilde{\eta}) \in I^2,$$

where $B_1(-t, t, t; \cdot)$ denotes the univariate B-spline of degree 1 with 3 knots $-t, t, t$ and normalized with $\int_I B_1(-t, t, t; u)\,du = 1/2$. In view of the interpolation conditions (2.10) and the above representations of a mixed divided difference we obtain

$$0 < [-t, t, t]_x\,[-t, t, t]_y\,\phi = \frac{\phi^{(1,1)}(t, t)}{4t^2}.$$

In a similar way we get

$$\phi^{(1,1)}(t, -t) < 0.$$

Therefore for the sign of $\phi^{(1,1)}$ on the diagonals $I_1 = \{(x, y) \in I^2 : x = y\}$ and $I_2 = \{(x, y) \in I^2 : x = -y\}$ (excluding the origin, where $\phi^{(1,1)}(0, 0) = 0$) we have

$$\operatorname{sign}[\phi^{(1,1)}_{|I_1}] = 1 \qquad \text{and} \qquad \operatorname{sign}[\phi^{(1,1)}_{|I_2}] = -1.$$

Further, differentiating the identities

$$\phi^{(1,0)}(x, x) = \phi^{(1,0)}(x, -x) = \phi^{(0,1)}(x, x) = \phi^{(0,1)}(x, -x) = 0, \qquad x \in I$$

we get

$$\operatorname{sign}[\phi^{(2,0)}_{|I_1}] = \operatorname{sign}[\phi^{(2,0)}_{|I_2}] = \operatorname{sign}[\phi^{(0,2)}_{|I_1}] = \operatorname{sign}[\phi^{(0,2)}_{|I_2}] = -1. \quad (2.11)$$

Assume for definiteness that $x_0 + y_0 < 0 < -x_0 + y_0$ (all other cases are similar to this one). In view of $\phi(x_0, y_0) = 0$, the interpolation conditions (2.10) and Rolle's theorem we obtain

$$\phi^{(2,0)}(\tilde{x}_0, y_0) = 0$$

for some $x_0 < \tilde{x}_0 < 0$ and this leads us to a contradiction with (2.11), observing that the function $\phi^{(2,0)}(\tilde{x}_0, y)$ is a strictly convex function with respect to y by assumption. Analogously, if we suppose that $D^{2,2}\phi < 0$ on I^2 we will get a contradiction by using the function $-\phi$ in a similar way.

Therefore, there exists a point $(\xi, \eta) \in I^2$ such that $D^{2,2}\phi(\xi, \eta) = 0$. In view of $D^{2,2}h_f^* \equiv 0$ we obtain (2.9). ∎

COROLLARY 2.4. *Let $f \in C^{2,2}(I^2)$ and let $D^{2,2}f \geqslant 0$ on I^2. Then we have*

$$f(x, y) \leqslant h_f^*(x, y) \qquad \text{for} \quad (x, y) \in I^2.$$

2.2. *Cubature Formula Based on the Interpolation Formula* (2.8)

In view of the fact that $D_{1,0}(t)$ and $D_{0,1}(t)$ (see representation (2.8) for h_f^*) are even functions (with respect to the variable $t \in [-1, 1]$) it is readily seen that

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} h_f^*(x, y)\,dx\,dy &= \int_{-1}^{1}\int_{0}^{x} [f^{(1,0)}(t, t) + f^{(0,1)}(t, t)]\,dt\,dx \\
&\quad + \int_{-1}^{1}\int_{0}^{x} [f^{(1,0)}(t, -t) - f^{(0,1)}(t, -t)]\,dt\,dx + 4f(0, 0) \\
&= \int_{-1}^{1}\int_{0}^{x} \frac{d}{dt}[f(t, t)]\,dt\,dx \\
&\quad + \int_{-1}^{1}\int_{0}^{x} \frac{d}{dt}[f(t, -t)]\,dt\,dx + 4f(0, 0) \\
&= \int_{-1}^{1} f(x, x)\,dx + \int_{-1}^{1} f(x, -x)\,dx.
\end{aligned}$$

From here we obtain the following cubature formula

$$\int_{I^2} f \approx \int_{I^2} h_f^* \quad \text{or} \quad \int_{I^2} f \approx \int_I f(x, x)\,dx + \int_I f(x, -x)\,dx. \tag{2.12}$$

By the uniqueness of the transfinite Hermite $B^{2,2}$-interpolant on $\times^*$ it follows that $h \equiv h_h^*$ for any $h \in B^{2,2}$. Hence, the cubature formula (2.12) is exact for all function from the linear space $B^{2,2}(I^2)$. The fact that the cubature (2.12) is exact in $B^{2,2}(I^2)$ could be easily derived from the representation (2.2) if the form of the cubature (2.12) was previously known. Let us denote

$$CF^*(f) := \int_I f(x, x)\,dx + \int_I f(x, -x)\,dx.$$

Our purpose now is to obtain an expression for the error

$$\mathscr{L}(f) := \int_{I^2} f - CF^*(f)$$

of the cubature (2.12).

Let $\tilde{h}_f(x, y)$ denote the transfinite Hermite $B^{2,2}(I^2)$-interpolant of $f \in C^{2,2}(I^2)$ based on the interpolation information

$$f(-1, y),\ f^{(1,0)}(-1, y),\ f(x, -1),\ f^{(0,1)}(x, -1).$$

The following representation holds [9]

$$f(x, y) - \tilde{h}_f(x, y) = \int_{I^2} (x-u)_+\ (y-v)_+\ D^{2,2} f(u, v)\, du\, dv. \tag{2.13}$$

Applying the linear functional $\mathscr{L}$ to the above identity we obtain Peano kernel representation for the error of the cubature (2.12).

THEOREM 2.5. *Let $f \in C^{2,2}(I^2)$ be a given function. Then we have*

$$\mathscr{L}(f) = \int_{I^2} f - CF^*(f) = \int_{I^2} K^*(u, v)\, D^{2,2} f(u, v)\, du\, dv,$$

where the Peano kernel K^* is given by

$$\begin{aligned} K^*(u, v) &:= \mathscr{L}[(\cdot - u)_+\ (\cdot - v)_+] \\ &= \frac{(1-u)^2\,(1-v)^2}{4} - \frac{(1-u)(1-v)^2}{2} \\ &\quad + \frac{(1-v)^3}{6} - \frac{(u-v)_+^3}{6} - \frac{(-u-v)_+^3}{6}. \end{aligned}$$

Moreover, $K^* \leqslant 0$ on I^2, and by the mean value theorem

$$\int_{I^2} f - CF^*(f) = -\tfrac{4}{45}\, D^{2,2} f(\xi, \eta), \tag{2.14}$$

where $(\xi, \eta) \in I^2$.

Proof. Direct calculations show that the Peano kernel

$$K^*(u, v) := \mathscr{L}[(\cdot - u)_+\ (\cdot - v)_+]$$

can be represented by the above expression. We shall prove that $K^* \leqslant 0$ on I^2. In view of the identity $(u-v)_+^3 = (u-v)^3 + (v-u)_+^3$ we have $K^*(u, v) = K^*(v, u)$, $(u, v) \in I^2$. Hence it is sufficient to check only the following two cases.

(i) Let $|u| \leqslant v$. Then

$$K^*(u, v) \leqslant -[(v+1/3)(1-v)^3]/4 \leqslant 0.$$

(ii) For $u \leqslant -|v|$ we have

$$K^*(u, v) \leqslant [(u+1)^2 \, (|v|-1)(|v|+1/3)]/4 \leqslant 0.$$

Therefore, $K^*(x, y) \leqslant 0$ on I^2 with equality only at the points $(\pm 1, v)$ and $(u, \pm 1)$. For $f_0(x, y) := x^2 y^2$ we have

$$\mathscr{L}(f_0) = \int_{I^2} f_0 - CF^*(f_0) = \int_{I^2} K^*(u, v) \, D^{2,2} f_0(u, v) \, du \, dv$$

$$= 4 \int_{I^2} K^*(u, v) \, du \, dv.$$

Simple calculations show that $\mathscr{L}(f_0) = -16/45$. Hence,

$$\int_{I^2} K^*(u, v) \, du \, dv = \tfrac{1}{4} \mathscr{L}(f_0) = -\tfrac{4}{45}.$$

By the mean value theorem we complete the proof of Theorem 2.5. ▮

2.3. *Best One-Sided L^1-Approximation from Above by $B^{2,2}$-Blending Functions*

Proof of Theorem 1.1. By Theorem 2.3 we have $h_f^* \geqslant f$ on I^2. Let $h \in B^{2,2}(I^2)$ and $h \geqslant f$ on I^2. We have

$$\int_{I^2} (h-f) = \int_{I^2} (h-h_f^*) + \int_{I^2} (h_f^* - f).$$

In view of the cubature (2.12), (2.14) we obtain

$$\int_{I^2} (h-h_f^*) = \int_{-1}^{1} (h-h_f^*)(x, x) \, dx + \int_{-1}^{1} (h-h_f^*)(x, -x) \, dx$$

$$= \int_{-1}^{1} (h-f)(x, x) \, dx + \int_{-1}^{1} (h-f)(x, -x) \, dx \geqslant 0,$$

since $h \geqslant f$ in I^2 and $h^*_{f|\times^*} = f_{|\times^*}$. From here

$$\int_{I^2} (h-f) \geqslant \int_{I^2} (h^*_f - f) \tag{2.15}$$

for each $h \in B^{2,2}(I^2)$ satisfying $h-f \geqslant 0$. Hence, h^*_f is a best one-sided L^1-approximant to f from above with respect to $B^{2,2}(I^2)$.

Next, we prove the uniqueness of the best one-sided approximant h^*_f. Assume that for some $h = h_0 \in B^{2,2}(I^2)$, $h_0 \geqslant f$ on I^2 we have equality in (2.15). Therefore,

$$\int_{I^2} (h_0 - h^*_f) = 0.$$

The cubature (2.14) yields

$$0 = \int_{I^2} (h^*_f - h_0) = \int_{-1}^{1} (h^*_f - h_0)(x, x)\, dx + \int_{-1}^{1} (h^*_f - h_0)(x, -x)\, dx \leqslant 0,$$

since $h_0 \geqslant f$ on I^2 and $h^*_{f|\times^*} = f_{|\times^*}$. Thus, we have

$$h_{0|\times^*} = h^*_{f|\times^*} = f_{|\times^*}$$

and consequently

$$\min_{I^2} (h_0 - f) = (h_0 - f)_{|\times^*} = 0.$$

This implies

$$\mathbf{grad}\, h_{0|\times^*} = \mathbf{grad}\, f_{|\times^*} = \mathbf{grad}\, h^*_{f|\times^*}.$$

In view of the uniqueness of the transfinite $B^{2,2}$-blending interpolant (2.1), (2.8) we obtain $h_0 \equiv h^*_f$. *Hence, h^*_f is the unique best one-sided from above L^1-approximant to f with respect to the infinite-dimensional linear space $B^{2,2}(I^2)$.* ∎

Remark. The first part of the above proof showing that h^*_f is a best one-sided L^1-approximant from above follows by [4, Proposition 1], too. Let us mention that in the case of harmonic approximation a part of this proposition is essentially given in [19]. In the particular case of blending approximation we arrived also at a part of the same proposition in our studies on best one-sided blending L^1-approximation, published in [9]. This interesting proposition is not a prescription for getting directly *canonical grid* multidimensional approximation results (see Proposition 3.4) but it shows that natural multidimensional extensions of the one-dimensional Markov's *canonical point set* theorem [1, pp. 82–85; 11, p. 87] are possible.

3. BEST ONE-SIDED L^1-APPROXIMATION FROM BELOW BY BLENDING FUNCTIONS OF ORDER (2, 2)

Let $f \in C^{2,2}(I^2)$ be a given function with $D^{2,2}f \geqslant 0$ on I^2. In addition, suppose that $f = f_0 + h$, where f_0 is even with respect to one of the variables and $h \in B^{2,2}(I^2)$. We shall prove that the unique best one-sided L^1-approximant to f from below is the unique solution h_*^f of the transfinite Hermite interpolation problem

$$h^f_{*|\Diamond_*} = f_{|\Diamond_*} \qquad \text{and} \qquad \mathbf{grad}\, h^f_{*|\Diamond_*} = \mathbf{grad}\, f_{|\Diamond_*} \tag{3.1}$$

from $B^{2,2}(I^2)$ on the canonical grid

$$\Diamond_* := \{(x, y) \in I^2 : |x| + |y| = 1\}.$$

Remark. Our expectation was that the best one-sided L^1-approximant from below should be given by blending interpolation on a fixed canonical point set for the entire cone class of $C^{2,2}(I^2)$-functions with non negative (2, 2) derivative on I^2. However, this is not the case. It is shown (Proposition 3.4, Section 3.3) that there is no canonical point set for the entire cone class under consideration when dealing with the best one-sided $L^1(I^2)$-approximation from below by $B^{2,2}(I^2)$.

3.1. *Transfinite Hermite Interpolation by $B^{2,2}(I^2)$-Blending Functions on $\Diamond_*$*

Let $f \in C^{2,2}(I^2)$. We denote by $f^{e,e}$, $f^{e,o}$, $f^{o,e}$, and $f^{o,o}$ its even-even, even-odd, odd-even, and odd-odd components, respectively. It is easily seen that

$$f^{e,e}(x, y) = \tfrac{1}{4}\,[f(x, y) + f(-x, y) + f(x, -y) + f(-x, -y)]$$

and analogous formulas for $f^{e,o}$, $f^{o,e}$ and $f^{o,o}$ hold. Obviously,

$$f = f^{e,e} + f^{e,o} + f^{o,e} + f^{o,o}.$$

For $f^{e,e} \in C^{2,2}(I^2)$ an explicit solution $h_*^{e,e}$ of the transfinite Hermite interpolation problem (3.1) can be given by the formula

$$h_*^{e,e}(x, y) := \int_{1/2}^{|x|} D^{1,0} f^{e,e}(t, 1-t)\, dt + \int_{1/2}^{|y|} D^{0,1} f^{e,e}(1-t, t)\, dt + f^{e,e}(1/2, 1/2). \tag{3.2}$$

Analogously, explicit solutions $h_*^{e,o}$, $h_*^{o,e}$ and $h_*^{o,o}$ of the interpolation problem (3.1) to $f^{e,o}$, $f^{o,e}$, and $f^{o,o}$, respectively can be given by the next formulas:

$$h_*^{e,o}(x,y) := f^{e,o}(1-|y|,y) + y\left\{\int_{1/2}^{|x|} \frac{D^{1,0}f^{e,o}(t,1-t)}{1-t}\,dt - \int_{1/2}^{1-|y|} \frac{D^{1,0}f^{e,o}(t,1-t)}{1-t}\,dt\right\}; \tag{3.3}$$

$$h_*^{o,e}(x,y) := f^{o,e}(x,1-|x|) + x\left\{\int_{1/2}^{|y|} \frac{D^{0,1}f^{o,e}(1-t,t)}{1-t}\,dt - \int_{1/2}^{1-|x|} \frac{D^{0,1}f^{o,e}(1-t,t)}{1-t}\,dt\right\}; \tag{3.4}$$

$$h_*^{o,o}(x,y) := xy\left\{\int_{1/2}^{|x|} \frac{tD^{1,0}f^{o,o}(t,1-t) - f^{o,o}(t,1-t)}{t^2(1-t)}\,dt + \int_{1/2}^{|y|} \frac{tD^{0,1}f^{o,o}(1-t,t) - f^{o,o}(1-t,t)}{t^2(1-t)}\,dt + 4f^{o,o}(1/2,1/2)\right\}. \tag{3.5}$$

Long, but simple and rewarding calculations based on L'Hospital rule show that:

(1) The functions $h_*^{e,e}$, $h_*^{e,o}$ $h_*^{o,e}$, and $h_*^{o,o}$ belong to $C^{2,2}(I^2)$. Hence, they are $B^{2,2}$-blending functions.

(2) Each of the functions $h_*^{e,e}$, $h_*^{e,o}$, $h_*^{o,e}$, and $h_*^{o,o}$ satisfies the transfinite interpolation conditions (3.1) with f replaced by $f^{e,e}$, $f^{e,o}$, $f^{o,e}$, and $f^{o,o}$, respectively.

The next theorem concerns the problem of transfinite Hermite interpolation by $B^{2,2}(I^2)$-functions on $\Diamond_*$.

Theorem 3.1. *Let $f \in C^{2,2}(I^2)$. Then we have:*

(a) *the function h_*^f defined by*

$$h_*^f := h_*^{e,e} + h_*^{e,o} + h_*^{o,e} + h_*^{o,o}$$

is the unique solution of the transfinite interpolation problem (3.1) *to f from $B^{2,2}(I^2)$;*

(b) *if f is even with respect to x or to y then the following error representation holds true*

$$f(x,y)-h_*^f(x,y)=\tfrac{1}{4}D^{2,2}f(\xi,\eta)(|x|+|y|-1)^2 \times[-\tfrac{1}{2}(|x|-|y|)^2+\tfrac{1}{3}(|x|+|y|)+\tfrac{1}{6}], \tag{3.6}$$

where $(\xi,\eta)=(\xi(x,y),\eta(x,y))\in I^2$.

Proof of (a). *Existence of a solution.* By using the fact that the transfinite Hermite interpolation problem (3.1) is a linear one and in view of the obvious equality

$$f=f^{e,e}+f^{e,o}+f^{o,e}+f^{o,o}$$

we arrive to the conclusion that h_*^f is a solution of the transfinite interpolation problem (3.1).

Uniqueness of the solution. Let $f\in C^{2,2}(I^2)$. In addition, let $f_{|\diamond_*}=0$ and let $\mathbf{grad}\, f_{|\diamond_*}=\mathbf{0}$. Then the same interpolation conditions hold for $f^{e,e}$, $f^{e,o}$, $f^{o,e}$, $f^{o,o}$ since the point set $\diamond_*$ is symmetric with respect to the coordinate axes. Suppose that $h(x,y)\in B^{2,2}(I^2)$ is *an arbitrary solution* to f of the interpolation problem (3.1) as above. The function h, being from $B^{2,2}(I^2)$, can be represented in a form $h(x,y)=a(y)\,x+b(y)+c(x)\,y+g(x)$ with $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $g(\cdot)$ from $C^2(I)$. Note, that *the transfinite Hermite interpolation* (3.1) *is a linear operator from* $C^{2,2}(I^2)$ *to* $B^{2,2}(I^2)$. In view of this, the $B^{2,2}$-blending functions $h^{e,e}$, $h^{e,o}$, $h^{o,e}$, $h^{o,o}$ satisfy the transfinite interpolation conditions (3.1) with f replaced by $f^{e,e}$, $f^{e,o}$, $f^{o,e}$, $f^{o,o}$, respectively. Note, that $h=h^{e,e}+h^{e,o}+h^{o,e}+h^{o,o}$. Simple verifications show

$$h^{e,e}(x,y)=1/2[(g(x)+g(-x))+(b(y)+b(-y))].$$

Hence, $h^{e,e}(x,y)$ has the form $h_1(x)+h_2(y)$ where h_1 and h_2 are even. Analogously, $h^{e,o}$, $h^{o,e}$, $h^{o,o}$ have representations $yh_3(x)+h_4(y)$, $h_5(x)+xh_6(y)$, $yh_7(x)+xh_8(y)$, respectively, where h_1, h_2, h_3, h_6 are even and h_4, h_5, h_7, h_8 are odd. In view of the interpolation conditions (3.1), it is easily seen that the even functions h_1, h_2, h_3, h_6 and the odd functions h_4, h_5, h_7, h_8 are solutions of a simple system of differential equations:

$$\begin{aligned}
&h_1'(x)=h_2'(x)=h_1(x)+h_2(1-x)=0, \qquad x\in[0,1];\\
&h_3'(x)=h_3(x)+h_4'(1-x)=(1-x)\,h_3(x)+h_4(1-x)=0, \qquad x\in[0,1];\\
&h_6'(x)=h_6(x)+h_5'(1-x)=(1-x)\,h_6(x)+h_5(1-x)=0, \qquad x\in[0,1];\\
&(1-x)\,h_7'(x)+h_8(1-x)=xh_8'(1-x)+h_7(x)\\
&(1-x)\,h_7(x)+xh_8(1-x)=0, \qquad x\in[0,1].
\end{aligned}$$

All solutions of the above system have to satisfy

$$h_1(x)+h_2(y)=yh_3(x)+h_4(y)$$
$$=h_5(x)+xh_6(y)=yh_7(x)+xh_8(y)=0, \qquad (x,y)\in I^2.$$

Hence, $h\equiv 0$ and the uniqueness follows. Part (a) of Theorem 3.1 is proved.
For the proof of part (b) we need the following auxiliary result.

LEMMA 3.2. *Let $\phi\in C^{2,2}(I^2)$ be given such that*

$$\phi_{|\Diamond_*}=0, \qquad \mathbf{grad}\ \phi_{|\Diamond_*}=\mathbf{0} \tag{3.7}$$

and

$$D^{2,2}\phi>0 \qquad on \quad I^2. \tag{3.8}$$

Then we have:

(i) $\phi(x,y)+\phi(-x,y)>0$ *for all* $(x,y)\in I^2\setminus\Diamond_*$.

(ii) *The function ϕ satisfies a minimum principle: ϕ cannot attain its minimum value on* $\overline{\mathbf{int}\,\Diamond_*}$ *at* $\mathbf{int}\,\Diamond_*$, *i.e.,* $\phi(x,y)>0$ *on* $\mathbf{int}\,\Diamond_*$.

Proof of Lemma (3.2). The crucial point is to consider a second order divided difference of $\phi(x,y)$ with respect to the variable x,

$$F_t(y)=[-t,t,t]\,\phi(\cdot,y)=\frac{1}{2t}\left[\phi^{(1,0)}(t,y)-\frac{\phi(t,y)-\phi(-t,y)}{2t}\right]$$

as a function of y for fixed $t\neq 0$. In view of (3.8) it is a strictly convex function with respect to y and by (3.7) it vanishes for all $(t,y)\in\Diamond_*$. Hence, for $(t,y)\in I^2$ with $t>0$ we have

$$\phi^{(1,0)}(-t,y)<\frac{\phi(t,y)-\phi(-t,y)}{2t}<\phi^{(1,0)}(t,y) \tag{3.9}$$

if $(t,y)\in I^2\setminus\overline{\mathbf{int}\,\Diamond_*}$ and

$$\phi^{(1,0)}(-t,y)>\frac{\phi(t,y)-\phi(-t,y)}{2t}>\phi^{(1,0)}(t,y) \tag{3.10}$$

if $(t,y)\in\mathbf{int}\,\Diamond_*$. Integrating (3.9) or (3.10) (the choice depends on the location of (x,y) on I^2) with respect to t from x to $1-|y|$ we obtain (i).

(ii) Suppose that there is a point $(x_m, y_m) \in \mathbf{int}\,\Diamond_*$ such that

$$\phi(x_m, y_m) = \min_{(x,y) \in \mathbf{int}\,\Diamond_*} \phi(x, y). \tag{3.11}$$

From here, $\mathbf{grad}\,\phi(x_m, y_m) = \mathbf{0}$. By (i) we have $\phi(0, y) > 0$ and $\phi(x, 0) > 0$. Thus, $x_m \neq 0$. Without any loss of generality, assume that $x_m > 0$. Then (3.10) with $t = x_m$, $y = y_m$ yields $\phi(-x_m, y_m) < \phi(x_m, y_m)$ and we are led to a contradiction with (3.11), since $(-x_m, y_m) \in \mathbf{int}\,\Diamond_*$. ▮

Remark. If in Lemma 3.2 we replace (3.8) by $D^{2,2}\phi \geqslant 0$ on I^2, then $\phi(x, y) + \phi(-x, y) \geqslant 0$ on I^2, which follows by a similar argument. Analogously, if $D^{2,2} f \geqslant 0$ on I^2 then

$$(f - h_*^f)(x, y) + (f - h_*^f)(x, -y) \geqslant 0 \qquad \text{for all} \quad (x, y) \in I^2.$$

Proof of part (b) *of Theorem* 3.1. Now we shall prove that the error representation (3.6) holds true for any function which is even with respect to one of the variables. Let $f(x, y) \in C^{2,2}(I^2)$ be an even function with respect to x, i.e., $f(x, y) = f(-x, y)$, $(x, y) \in I^2$. Let h_*^f be the unique $B^{2,2}(I^2)$-interpolant satisfying the interpolation conditions (3.1). Since f is even with respect to x we have $h_*^{o,e} = h_*^{o,o} = 0$. Let us denote

$$\begin{aligned} g(x, y) &:= x^2 y^2 - \tfrac{1}{2}(x^4 + y^4) + \tfrac{4}{3}(|x|^3 + |y|^3) - (x^2 + y^2) + \tfrac{1}{6} \\ &= (|x| + |y| - 1)^2 \left(-\tfrac{1}{2}(|x| - |y|)^2 + \tfrac{1}{3}(|x| + |y|) + \tfrac{1}{6}\right) \\ &=: (|x| + |y| - 1)^2\, g_1(x, y). \end{aligned} \tag{3.12}$$

It can be easily seen that $g_1 \geqslant 0$ on I^2 with equality only at the points $(0, \pm 1)$ and $(\pm 1, 0)$. Let $(x_0, y_0) \in I^2 \setminus \Diamond_*$. Consider the auxiliary function (which is even with respect to x)

$$\phi(x, y) := f(x, y) - h_*^f(x, y) - \frac{f(x_0, y_0) - h_*^f(x_0, y_0)}{g(x_0, y_0)}\, g(x, y).$$

It is clear that ϕ satisfies (3.7) and $\phi(x_0, y_0) = 0$. Suppose that $D^{2,2}\phi > 0$ on I^2. Then Lemma 3.2(i) yields

$$\phi(x, y) = \tfrac{1}{2}[\phi(x, y) + \phi(-x, y)] > 0 \qquad \text{for all} \quad (x, y) \in I^2 \setminus \Diamond_*,$$

and we are led to a contradiction with $\phi(x_0, y_0) = 0$. Suppose that $D^{2,2}\phi < 0$ on I^2. Then we consider the function $-\phi$ and again we arrive to a contradiction. Thus, there exists a point $(\xi, \eta) = (\xi(x_0, y_0), \eta(x_0, y_0))$ such that $D^{2,2}\phi(\xi, \eta) = 0$ and we end the proof. ▮

Remark. The transfinite $B^{2,2}(I^2)$-blending interpolant h_*^f is *a* $B^{2,2}(I^2)$-*spline function with two line segment knots* $\{(x, 0)\colon -1 \leqslant x \leqslant 1\}$ *and* $\{(0, y)\colon -1 \leqslant y \leqslant 1\}$. In general it is only a $C^{2,2}$-function on I^2 (see the Example in Section 1), while on Ω_j, $j = 1, ..., 4$ (see Section 2.1) it has the same smoothness as f.

3.2. *Cubature Formula Based on the* $(B^{2,2}(I^2), \diamondsuit_*)$ *Interpolation*

Integrating the $B^{2,2}(I^2)$-interpolant h_*^f (explicitly given by (3.2)–(3.5)) over I^2 we obtain the cubature formula

$$\int_{I^2} f \approx \int_I f(x, 1-|x|)\, dx + \int_I f(x, |x|-1)\, dx, \tag{3.13}$$

which is exact for all blending-functions in $B^{2,2}(I^2)$. In contrast with the error representation (3.6) the next representation for the error of the cubature (3.13) holds true for any function $f \in C^{2,2}(I^2)$. Let us denote

$$CF_*(f) := \int_I f(x, 1-|x|)\, dx + \int_I f(x, |x|-1)\, dx.$$

THEOREM 3.3. *Let $f \in C^{2,2}(I^2)$. Then there exists a point $(\rho, \sigma) \in I^2$ such that*

$$\int_{I^2} f - CF_*(f) = \tfrac{7}{90} D^{(2,2)} f(\rho, \sigma). \tag{3.14}$$

Proof. By Lemma 3.2(i) we have

$$\int_{I^2} \phi = \int_{-1}^{1} \int_{-1}^{1} \tfrac{1}{2} [\phi(x, y) + \phi(-x, y)]\, dx\, dy > 0 \tag{3.15}$$

for any $\phi \in C^{2,2}(I^2)$ satisfying (3.7) and (3.8). Let us consider the auxiliary function

$$\phi_*(x, y) := f(x, y) - h_*^f(x, y) - \frac{\int_{I^2} f - \int_{I^2} h_*^f}{\int_{I^2} g}\, g(x, y),$$

where g is defined by (3.12). We apply the technique described in the proof of Theorem 3.1, part (b) to the function ϕ_*. Suppose that $D^{2,2}\phi_* > 0$ or $D^{2,2}\phi_* < 0$ on I^2. Then (3.15) with ϕ_* or $-\phi_*$ will contradict to the fact that $\int_{I^2} \phi_* = 0$. Therefore, there exists a point $(\rho, \sigma) \in I^2$ such that $D^{2,2}\phi_*(\rho, \sigma) = 0$. In view of $\int_{I^2} g = (14)/(45)$, $\int_{I^2} h_*^f = CF_*(f)$, and $D^{2,2} g = 4$ we complete the proof. ∎

3.3. *One-Sided Approximation from Below by $B^{2,2}(I^2)$-Functions*

Proof of Theorem 1.2. By Theorem 3.1 (b) we have $h_*^f \leqslant f$ on I^2. Since the cubature formula (3.14) is exact for all functions from $B^{2,2}(I^2)$, it follows, by analogy with the proof of Theorem 1.1, that h_*^f is a best one-sided approximant from below to f. Hence,

$$\int_{I^2} (f-h) \geqslant \int_{I^2} (f-h_*^f) \tag{3.16}$$

for each $h \in B^{2,2}(I^2)$ satisfying $h \leqslant f$ on I^2.

Next, we prove the uniqueness of the best one-sided approximant h_*^f. Let us assume that for some $h_0 \in B^{2,2}(I^2)$, $h_0 \leqslant f$ on I^2 we have equality in (3.16). Therefore,

$$\int_{I^2} (h_0 - h_*^f) = 0.$$

In view of the cubature (3.14), which is exact in $B^{2,2}(I^2)$, we obtain

$$0 = \int_{I^2} (h_*^f - h_0) = \int_{-1}^{1} (h_*^f - h_0)(x, 1-|x|)\, dx$$
$$+ \int_{-1}^{1} (h_*^f - h_0)(x, |x|-1)\, dx \geqslant 0,$$

taking into account that $h_0 \leqslant f$ on I^2 and $h^f_{*|\Diamond_*} = f_{|\Diamond_*}$. Thus,

$$h_{0|\Diamond_*} = h_{*|\Diamond_*} = f_{|\Diamond_*}$$

and consequently

$$\min_{I^2} (f-h_0) = (f-h_0)_{|\Diamond_*} = 0.$$

This implies that

$$\mathbf{grad}\, h_{0|\Diamond_*} = \mathbf{grad}\, f_{|\Diamond_*} = \mathbf{grad}\, h_{*|\Diamond_*}$$

and by the uniqueness of the solution of the interpolation problem (3.1) we get $h_0 \equiv h_*^f$. Hence, $h_*^f \in B^{2,2}(I^2)$ is the unique best one-sided L^1-approximant to f from below. ∎

PROPOSITION 3.4. *There is no subset of I^2, that is a canonical point set for the problem of best one-sided $L^1(I^2)$-approximation from below by $B^{2,2}(I^2)$-functions to the cone of all $C^{2,2}(I^2)$-functions with non-negative* (2, 2) *mixed derivative.*

Proof. We have proved in Theorem 1.2 that $\Diamond_*$ is a canonical point set for the class of all $C^{2,2}(I^2)$-functions which are even with respect to one of its variables with non-negative (2, 2) derivative. Thus, it is sufficient to show, that there is a function $\psi \in C^{2,2}(I^2)$ with non-negative (2, 2) derivative, such that its transfinite Hermite $B^{2,2}(I^2)$-blending interpolant on $\Diamond_*$ is not a best one-sided L^1-approximant to ψ from below.

Consider the function $\psi \in C^{2,2}(I^2)$ defined by

$$\psi(x, y) := (x^3 - |x|^3)(y^3 - |y|^3), \qquad (x, y) \in I^2.$$

We have

$$D^{2,2}\psi(x, y) = 36(x - |x|)(y - |y|) \geqslant 0, \qquad (x, y) \in I^2.$$

We get from (3.2)–(3.5) for the components of h_*^ψ

$$h_*^{e,e} = \int_{1/2}^{|x|} 3t^2(1-t)^3\,dt + \int_{1/2}^{|y|} 3t^2(1-t)^3\,dt + \tfrac{1}{64},$$

$$h_*^{e,o} = -(1-|y|)^3\,y^3 - y\left[\int_{1/2}^{|x|} 3t^2(1-t)^2\,dt - \int_{1/2}^{1-|y|} 3t^2(1-t)^2\,dt\right],$$

$$h_*^{o,e} = -(1-|x|)^3 x^3 - x\left[\int_{1/2}^{|y|} 3t^2(1-t)^2\,dt - \int_{1/2}^{1-|x|} 3t^2(1-t)^2\,dt\right],$$

$$h_*^{o,o} = xy\left[\int_{1/2}^{|x|} 2t(1-t)^2\,dt + \int_{1/2}^{|y|} 2t(1-t)^2\,dt + \tfrac{1}{16}\right].$$

It is not difficult to compute that

$$\psi(1, 1) - h_*^\psi(1, 1) = -\tfrac{1}{60}$$

and from here h_*^ψ does not belong to the cone $\{h \in B^{2,2}(I^2) : h \leqslant \psi\}$. Hence, h_*^ψ is not a best one-sided L^1-approximant to ψ from below. ▮

Remark. Following [4, Proposition 1] and taking into account Theorem 3.1 and Theorem 3.3 we can claim that h_*^ψ is a best one-sided L^1-approximant to ψ from below if $\psi \geqslant h_*^\psi$ on I^2. However, as we have seen, $\psi(x, y) < h_*^\psi(x, y)$ in a neighborhood of the point (1,1) in I^2.

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